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Abstract. I model two ways climate tail risk could threaten the resources available for consumption in an otherwise standard cake-eating problem. I show that precautionary behaviour is optimal, no matter how low the probability of catastrophic climate outcomes.

JEL Codes: E20, Q50.

Keywords: Climate change, Tail Risk, Tipping Point.

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Résumé Non Technique

Depuis longtemps, notre demande de biens et services produits par la nature dépasse la capacité de celle-ci à les fournir, entraînant une dégradation massive de la biosphère, c'est-à-dire de la somme de tous ses écosystèmes.

Cette situation a de nombreux effets néfastes, allant du réchauffement planétaire à l'extinction massive des espèces, du stress hydrique à des chocs météorologiques plus fréquents et plus graves. Ces dommages environnementaux sont irréversibles, du moins à l'échelle humaine.

Ce papier développe un modèle simple afin de démontrer qu'une plus grande fréquence d'événements naturels dévastateurs, c'est-à-dire d'épisodes qui causeraient des pertes irréversibles pour la biosphère, augmente la valeur sociale des ressources naturelles. Il en résulte que la politique optimale comporte une réduction de notre consommation de ces ressources.

La logique est simple : la possibilité d'événements naturels dévastateurs, aussi faible que soit leur probabilité, justifie un comportement de précaution. En d'autres termes, elle incite le décideur à accumuler une réserve de capital naturel qui permet de réduire la probabilité de chocs futurs et d'en amortir les effets s'ils se produisent.

What's done cannot be undone.

William Shakespeare

1. INTRODUCTION

Climate change increases the frequency and severity of weather shocks. I will argue that this heightened risk of devastating natural events raises the value of natural capital, making it optimal to lower our consumption of Nature's goods and services.

In this paper, I embed tail risk into the well-known "cake-eating" problem developed by Hotelling (1931). The economy starts with a finite stock of a homogeneous consumption good. As in Dasgupta and Heal (1974), there is no production, but a flow of the consumption good is fed into the economy at each instant. The central planner chooses the rate of consumption to maximise expected welfare.

I study two versions of the model. In the first version, a single tipping point threatens the economy. If it is crossed, the flow of the consumption good collapses, and the economy has no additional resources going forward.

In the second version of the model, it is the stock of the consumption good itself that is subject to both Brownian movement and to an unbounded number of stochastic jumps.

Though the two versions share the same mathematical structure, there is a key difference. In the first version, the planner knows what lies beyond the tipping point, and controls the likelihood of crossing it. In the second version, the planner cannot control the stochastic properties of the shocks, and his or her stock of the consumption good is threatened by much greater risks.[1]

The two theoretical frameworks convey the same message: the possibility of disastrous outcomes, however low their probability, motivates precautionary behaviour. That is, it becomes optimal to cut consumption to build a buffer of the consumption good.

My work is inspired by Weitzman (2011), Stern (2013) and Pindyck (2021), who stress the need to account for the uncertainty and nonlinearities inherent in the climate system. My paper also follows the call by Dasgupta (2021) to allow Nature to play a more prominent

[1]Think of the first version of the model as an environment where limiting the negative effects of climate change is still within reach. In contrast, think of the second version as an environment where the Earth is already beyond repair, so the human population faces much larger risks.

role in macroeconomic models. He stresses that if our demand for Nature's goods and services persistently exceeds Nature's ability to supply them, then the stock of natural capital deteriorates, bringing the biosphere closer to the brink of collapse.

In addition, my paper connects with the literature on economic growth and the environment [see Xepapadeas (2005) and references]. This body of work focuses on the gap between market outcomes and the social optimum in deterministic settings. Instead, my focus is on how the risk of catastrophic climate outcomes shapes the central planner's choices.

Lastly, my work is linked to the literature on optimal stopping problems in environmental economics [Pindyck (2000); Pindyck (2002)]. As in these studies, I draw attention to the policy implications of irreversible environmental damages. However, I focus on stochastic control problems where the control variables are adjusted continuously over time, rather than the optimal timing of a discrete policy action.

2. Model 1

2.1. **Set up.** Consider the following control system:

$$\begin{aligned} &\dot{s}(t) = m(t) - c(t), \quad t \in [0, T], \quad s(0) = s_0, \\ &s(t) \in \mathbb{R}_{>+} \forall t, \quad c(t) \in \mathbb{R}_{>+} \forall t, \end{aligned} \tag{1}$$

where $T < \infty$. Here $s(t)$ is the stock of a homogeneous consumption good; $c(t)$ is the rate of consumption; and $m(t)$ is the flow of this consumption good that is fed into the economy. At a certain stochastic time τ, the rate $m(t)$ jumps according to:

$$m(t) = \begin{cases} m & t < \tau, \\ 0 & t \geq \tau, \end{cases} \tag{2}$$

where $m > 0$. In words, the economy *may* cross a tipping point beyond which the flow of the consumption good collapses to zero. The word "may" is important here. The random variable τ takes values in $(0, \infty)$, while t takes values in $[0, T]$. Therefore there is no guarantee that the tipping point will be crossed in $[0, T]$.

The random variable τ is exponentially distributed with hazard rate $\lambda(s) \geq 0 \ \forall s \in \mathbb{R}_{>+}$. The latter rate is a C^1-function of s with $\lambda'(\cdot) \leq 0$ and $\sup_s \lambda(s) < \infty$. The exponential distribution is a natural choice, for it is often used to measure the expected time for an event to occur.[2]

Social welfare in $[0, T]$ is given by:

$$\int_0^T U(c(t))dt + B(s(T)). \tag{3}$$

$U(\cdot)$ is monotonically increasing, strictly concave, and twice differentiable everywhere. Also, $\lim_{c\to 0} U'(c) = \infty$. In addition, $B(\cdot)$, which is a C^2-function of $s(T)$ with $B'(\cdot) > 0$, measures the scrap value of the stock of the consumption good at T. I also assume that $\lim_{s\to 0} B'(s) = \infty$.[3]

The central planner chooses the consumption path, $\{c(t)\}_0^T$, that maximises (3) s.t. (1), (2).

Remark I. Think of $s(t)$ as the stock of natural capital, and of $m(t)$ as its regeneration rate - or equivalently, its yield per unit of time. Furthermore, think of $c(t)$ as the rate at which the human population harvest Nature's goods and services. At each point in time in this framework, our demand for these goods and services may differ from Nature's ability to supply them; the difference is accommodated by a change in $s(t)$. However, as the stock of natural capital is drawn down, Nature gets closer to the brink of collapse.

Remark II. I implicitly set the discount factor to 1. This assumption, made for mathematical tractability, may seem too strong at first. However, as will become clear, it is not driving the main insights of the paper. Furthermore, the principles of discounting set out in the 70s and 80s [e.g. Arrow and Lind (1970) and Dreze and Stern (1990)] can justify a discount factor of 1. Think of the discount factor between time 0 and time t as the shadow value of the consumption good at time t relative to time 0. This shadow value will depend on the state of affairs at these two dates. If the consumption good were to become scarce at time t (for instance, due to the crossing of a tipping point), then its shadow price would be high, and the discount factor could be close to unity (or even above it) [Stern (2013)].

2.2. **Optimality Conditions.** The above model is a piecewise deterministic optimal control problem. That is, the control system is governed by deterministic ordinary differential equations but shocked by a stochastic disturbance. I solve this problem using the extremal method

[2]Though the main insights of the paper do not depend on this assumption, the exponential distribution does make solving the model easier.

[3]This assumption ensures that $s(T) > 0$. In turn, this implies that $s(t) > 0$ for all $t \in [0, T]$, thus satisfying (1).

due to Seierstad (2009). For problems with a finite time horizon, this method, which is closely related to the Hamilton-Jacobi-Bellman (HJB) equation, yields a convenient backward induction solution procedure. Let the subscript a (b) denote the path of a given variable after (before) the tipping point is crossed.

2.2.1. *Dynamics after the jump.* Suppose the jump in $m(t)$ has already occurred; that is, $t > \tau$. The dynamics of the control system are given by:[4]

$$\begin{cases} \dot{q}^a(t) = 0, \\ \dot{s}^a(t) = -c^a(t), \\ \dot{\mu}^a(t) = -U(c^a(t)), \\ c^a(t) = U'^{-1}(q^a(t)), \end{cases} \tag{4}$$

with boundary conditions: $q^a(T) = B'(s^a(T))$, $\mu^a(T) = B(s^a(T))$, and $s^a(\tau) = s^b(\tau)$. Here $q(t)$ is a standard co-state variable measuring the value of an infinitesimal increase in $s(t)$. Think of $q(t)$ as a shadow price. In addition, the auxiliary state variable $\mu(t)$ measures the expected flow of remaining social welfare along the optimal path.

Three features are worth noting in system (4). First, $q^a(t)$, and hence $c^a(t)$, is constant. After crossing the tipping point, there is no more risk, and total consumption smoothing becomes optimal.[5] Second, the added satisfaction that the planner gets from one extra unit of s at time T pins down $q^a(t)$. This bequest motive, together with the curvature of the utility function, also determines $c^a(t)$. Third, once the tipping point is crossed, $s^a(t)$ declines steadily. Indeed, with a strictly positive rate of consumption and no new flow of the consumption good, $s^a(t)$ can only decline.

System (4) has the following closed-form solution:

$$\begin{cases} q^a(t) = B'(s^a(T)), \\ s^a(t) = s^b(\tau) + U'^{-1}(B'(s^a(T)))(\tau - t), \\ \mu^a(t) = B(s^a(T)) + U(U'^{-1}(q^a(t)))(T - t), \\ c^a(t) = U'^{-1}(B'(s^a(T))), \end{cases} \tag{5}$$

[4]In the interest of space, I directly report the optimality conditions of the model. Please refer to Naevdal (2006) and Seierstad (2009) for a detailed discussion on how to derive these optimality conditions in a general piecewise deterministic optimal control problem.

[5]As mentioned earlier, the discount rate is 0, so the planner values all periods equally.

where $s^a(T)$ is implicitly given by the second equation in (5).

2.2.2. *Dynamics before the jump.* Suppose now that the jump has not yet happened; that is, $t < \tau$. The dynamics of the system are given by:

$$\begin{cases} \dot{q}^b(t) = \lambda(s)[q^b(t) - q^a(t)] + \lambda'(s)[\mu^b(t) - \mu^a(t)], \\ \dot{s}^b(t) = m - c^b(t), \\ \dot{\mu}^b(t) = -U(c^b(t)) + \lambda(s)[\mu^b(t) - \mu^a(t)], \\ c^b(t) = U'^{-1}(q^b(t)), \end{cases} \tag{6}$$

with boundary conditions: $q^b(T) = B'(s^b(T))$, $\mu^b(T) = B(s^b(T))$, and $s^b(0) = s_0$.

In system (6), the planner accounts for the risk of crossing the tipping point: the optimal path *after* the jump affects the system dynamics *before* such the jump occurs. The planner also internalises how his or her choices affect the likelihood of crossing the tipping point. Indeed, both $\lambda(s)$ and $\lambda'(s)$ shape system (6).

Crucially, the planner's optimal choices include a precautionary component. As seen in the first equation of system (6), two factors motivate precautionary behaviour. Focus first on the term $\eta_1 = \lambda(s)[q^b(t) - q^a(t)]$. Once a jump happens, $s(t)$ becomes scarcer, and its shadow price, $q(t)$, jumps upwards. Hence, $q^b(t) - q^a(t) < 0 \ \forall t \in [0, \tau]$. Therefore, η_1 pushes $q^b(t)$ downwards over time. Put differently, η_1 calls for a low initial consumption rate that gradually increases over time. These dynamics allow the planner to build a cushion of $s(t)$ at the beginning of the time horizon. The goal is to limit the fall in consumption if the tipping point is ever crossed. The hazard rate, $\lambda(s)$, governs how significant this effect is.

Let us now look at the term $\eta_2 = \lambda'(s)[\mu^b(t) - \mu^a(t)]$. Once a jump happens and $s(t)$ becomes scarcer, the expected flow of remaining welfare, $\mu(t)$, jumps downwards. Hence, $\mu^b(t) - \mu^a(t) > 0 \ \forall t \in [0, \tau]$. Since $\lambda'(s) < 0$, η_2 pushes $q^b(t)$ downwards over time too. These downward dynamics generate a rising consumption profile, which, again, allows the planner to start by building a buffer of $s(t)$. However, η_2 captures a different goal from η_1: that of lowering the likelihood of crossing the tipping point.[6]

In sum, the risk of a tipping point generates two sources of precautionary behaviour and encourages the central planner to create a natural capital buffer.

[6] If the hazard rate was exogenous (i.e. $\lambda'(s) = 0$), η_2 would disappear.

2.3. **Numerical Examples.** The nonlinear boundary value problem (6) has no closed-form solution. Hence, I solve it using the collocation method proposed by Shampine et al. (2003). The numerical examples illustrate my key message - the possibility of crossing a tipping point encourages precautionary behaviour.

2.3.1. *Parametrisation.* I use the following functional forms: $U(\cdot) = \log[c(t)]$, $B(\cdot) = \phi\log[s(T)]$, and $\lambda(\cdot) = \gamma/s(t)$. Here $\phi > 0$ governs the size of the bequest motive, and $\gamma \geq 0$ determines the likelihood of crossing the tipping point and how sensitive this likelihood is to the planner's choices.

Turning to the parameter values, I think of the horizon $[0, T]$ as the period a policy-maker is in office. I thus set $T = 16$, which corresponds to 4 years at quarterly frequency. I also set $s_0 = 1$ and $m = 0.0125$. Hence, if s is depleted before the jump, it will take 20 years of zero-consumption (before the jump) to bring it back to its initial state.[7] Lastly, I do not choose a single value for γ and ϕ, because I have no empirical basis to do so. Instead, I perform comparative statics to assess how the model insights depend on these two parameters.

2.3.2. *Model Dynamics.* Figure 1 illustrates the dynamics described in 2.2. Solid blue lines show the path of the control system when the tipping point is crossed at $t = 10$. Dashed black lines, which serve as a benchmark, show the path of an hypothetical control system where there is no tipping point risk (i.e. $\lambda(s) = 0$ for all $s \in \mathbb{R}_{>+}$).[8] In this riskless benchmark, both the shadow price of capital and the rate of consumption are constant.

As discussed in 2.2.2, the possibility of crossing a tipping point at some uncertain future date encourages the planner to cut consumption in the initial stages of the simulation to boost the stock of natural capital. It is in this sense that the planner's optimal behaviour is precautionary.

2.3.3. *Comparative statics:* $\{\gamma, \phi\}$. Figure 2 plots the initial rate of consumption, $c(0)$, against the expected probability of crossing the tipping point in $(0, T]$, which I denote by $P^0(\tau < T)$. I vary this probability by changing γ.

[7] Nature only regenerates slowly. For example, Archer (2005) finds a mean atmospheric lifetime of anthropogenic CO_2 of 300 years. Likewise, Liebsch et al. (2008) find that the Brazilian Atlantic forest would take at least 65 years to partly regenerate, and up to 4,000 years to fully regain its pristine state. Along the same lines, Duarte et al. (2020) suggest that substantial recovery of the abundance, structure and function of marine life would need 30 years.

[8] In Figure 1, I set $\gamma = 0.08$, yielding an unconditional probability of crossing the tipping point in $[0, T]$ of 50%. I also set $\phi = 40$, so that the economy without tail risks features $c/m = 1.7$. The qualitative features of Figure 1 do not depend on these values.

FIGURE 1. Model dynamics.

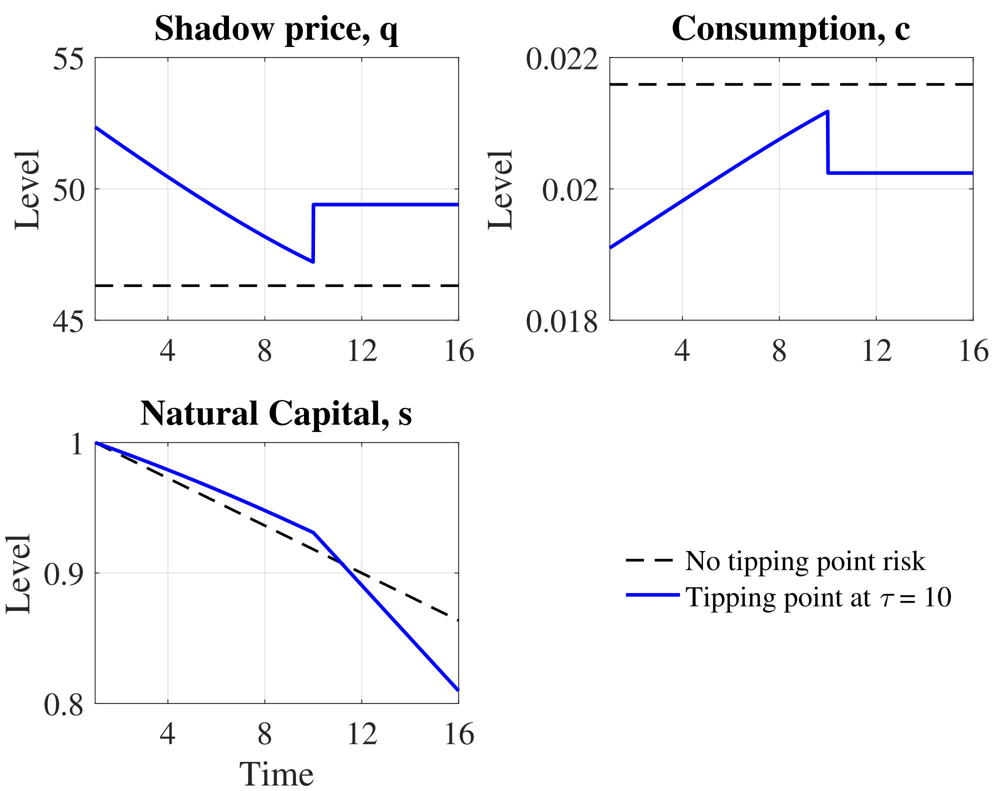
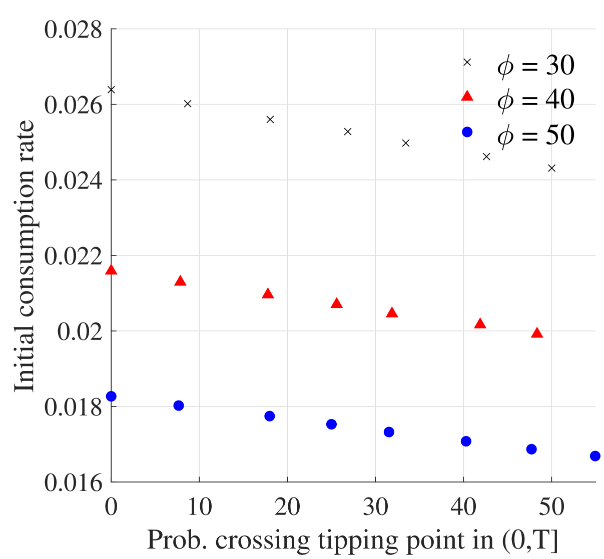


FIGURE 2. Comparative statics.



Raising $P^0(\tau < T)$ lowers $c(0)$, for a higher risk of crossing the tipping point boosts precautionary behaviour. Algebraically, increasing γ raises the absolute value of both $\lambda(s)$ and $\lambda'(s)$, thus pushing up $q(0)$ and pushing down $c(0)$.

I now assess the role played by the bequest motive (i.e. parameter ϕ). Increasing ϕ reduces $c(0)$. A stronger bequest motive raises the cost of crossing the tipping point, since this lowers $s(T)$. Therefore, a stronger bequest motive foster precautionary behaviour, which (again) increases $q(0)$ and decreases $c(0)$.

3. Model 2

In the first version of the model, the planner knew what lay beyond the tipping point and controlled the likelihood of crossing it. I now introduce a second version where the planner does not control the statistical properties of the shocks and faces much greater risks. Specifically, the stock of consumption good itself will be subject to Brownian movement and to an unbounded number of stochastic jumps.

3.1. **Set up.**

Natural capital, $s(t) \in \mathbb{R}_{>+}$, is now governed by the following stochastic differential equation with jumps:

$$\begin{aligned} ds(t) &= [m - \hat{c}(t)]\, s(t)dt + \sigma s(t)dW(t) \\ s(\tau_i+) &= \mu_i s(\tau_i-). \end{aligned} \tag{7}$$

Here $m, \sigma > 0$ are constants, $\hat{c} = c/s$ is the consumption to capital ratio, and $W(t)$ is a 1-dimension standard Brownian motion process.[9] The μ_i's are i.i.d. random variables with density function $g(\mu)$ on $(0, 1)$. Note then that $\mathrm{E}\mu^{\zeta} > 1,\ \forall \zeta < 0$.

In English, $s(t)$ jumps at certain random times, τ_i. Between jumps, $s(t)$ follows a standard stochastic differential equation. The sizes of these jumps have a random component, $\mu_i \in (0, 1)$, which makes forecasting the future path of $s(t)$ harder.

The number of jumps, τ_i, is unbounded. The first jump point, τ_1, is exponentially distributed in $[0, \infty)$, while all subsequent jump points, τ_i for $i > 2$, are exponentially distributed in $[\tau_{i-1}, \infty)$. These exponential distributions share the same hazard rate, λ.

All sources of randomness are mutually independent. The solution to eq. (7) is piecewise continuous, and continuous in (τ_i, τ_{i+1}).

[9] Both Nature's regenerative rate and the diffusion coefficient are proportional to $s(t)$. This proportionality yields a closed-form solution of the model and ensures that $s(t) \in \mathbb{R}_{>+} \forall t$.

Since the central planner maximises social welfare in $[0, T]$, the value function is given by:

$$\begin{aligned} v(t,s) &= \sup_{c>0} \mathrm{E}\left[\int_0^T U(c(t))dt + B(s(T))\right], \\ (t,s) &\in [0,T] \times \mathbb{R}_{>+}. \end{aligned} \tag{8}$$

The functions $U(\cdot)$ and $B(\cdot)$ satisfy the usual conditions defined in 2.1.

3.2. **Solution.** The above model is a stochastic control problem with jumps. The value function, $v(t,s)$, satisfies the following HJB equation:[10]

$$\begin{aligned} &v_t(t,s) + \max_{c>0}\{U(c) + v_s(t,s)\left[m - \hat{c}(t)\right]s(t)\} \\ &+ \frac{1}{2}\sigma^2 s^2 v_{ss}(t,s) + \lambda \mathrm{E}\left[v(t,\mu s) - v(t,s)\right] = 0, \end{aligned} \tag{9}$$

with $v(T,s) = B(s(T))$.[11]

To obtain a closed-form smooth solution to eq. (9), $U(\cdot)$ and $B(\cdot)$ are specified with a constant risk aversion form and share the same exponent: $U(\cdot) = \frac{c(t)^\alpha}{\alpha}$, $B(\cdot) = \phi\frac{s(T)^\alpha}{\alpha}$, where $\phi > 0$. For simplicity, I set $\alpha < 0$, so that the degree of risk aversion is strictly larger than 1.

I look for a solution to the HJB equation of the form $\omega(t,s) = \theta(t)\frac{s(t)^\alpha}{\alpha}$ for some positive function $\theta(t)$. The boundary condition $v(T,s) = B(s(T))$ then becomes $\theta(T) = \phi$.

Solving the max operator in eq. (9) yields:

$$\hat{c}(t) = \theta(t)^{\frac{1}{\alpha-1}}.$$

Substituting this optimality condition together with $\omega(t,s)$ and its derivatives in the HJB equation gives:

$$\theta'(t) + (1-\alpha)\theta(t)^{\frac{\alpha}{\alpha-1}} + \gamma\theta(t) = 0, \tag{10}$$

where $\gamma = \alpha\{m + \frac{\alpha-1}{2}\sigma^2 + \frac{\lambda}{\alpha}\left[\mathrm{E}\mu^\alpha - 1\right]\}$.

Eq. (10) is a Bernoulli differential equation. Under the boundary condition $\theta(T) = \phi$, the particular solution to this Bernoulli differential equation is:

$$\theta(t) = \left[\frac{\alpha-1}{\gamma} + \left[\phi^{\frac{1}{1-\alpha}} + \frac{1-\alpha}{\gamma}\right]e^{\frac{\gamma}{1-\alpha}(T-t)}\right]^{1-\alpha}.$$

It can be easily shown that, as initially guessed, $\theta(t)$ is a positive function in $[0, T]$.

[10] See chapter 4 in Seierstad (2009) for a detailed discussion on solving stochastic control problems with jumps.

[11] Without jumps (i.e. $\lambda = 0$), eq. (9) is the standard HJB equation for diffusion processes.

The function $\omega(t,s) = \theta(t)\frac{s(t)^\alpha}{\alpha}$ is a C^2 function in $[0,T] \times \mathbb{R}_{>+}$, and satisfies the HJB equation (9) in $[0,T] \times \mathbb{R}_{>+}$ with the boundary condition $\omega(t,s) = \theta(t)\frac{s(t)^\alpha}{\alpha}$. Furthermore, for every pair $(t,s) \in [0,T] \times \mathbb{R}_{>+}$,

$$c^*(t) = \theta(t)^{\frac{1}{\alpha-1}} s(t),$$

solves the max operator in the HJB equation. Then $c^*(t)$ is optimal and $\omega(t,s) = v(t,s)$.

3.3. **Result.** I now state the model's key message:

Proposition 1. For any $s(t) \in \mathbb{R}_{>+}$ and $\gamma \neq 0$,

$$\begin{aligned} \frac{\partial v(t,s)}{\partial \sigma} < 0, \quad \frac{\partial v(t,s)}{\partial \lambda} < 0, \\ \frac{\partial c^*(t)}{\partial \sigma} < 0, \quad \frac{\partial c^*(t)}{\partial \lambda} < 0. \end{aligned} \tag{11}$$

Proof. See Appendix A.

In words: a more volatile Brownian motion (σ) lowers welfare, because agents are risk-averse. It also reduces consumption by inciting the planner to accumulate a natural capital buffer to cushion the effects of future shocks.

Likewise, the parameter determining the frequency of jumps (λ) also reduces welfare. Since the $\mu_i' s$ are drawn from a density function on $(0,1)$, jumps inevitably reduce the amount of resources available for consumption. Put differently, jumps entail a negative wealth effect. Crucially, the hazard rate λ also lowers consumption. Again, the planner builds a capital buffer to lower the likelihood that the stock of capital will get dangerously close to 0.

In sum, the threat posed by these two shocks reduces welfare and fosters precautionary behaviour.

Remark IV. Suppose that the $\mu_i' s$ are drawn from a Beta distribution with shape parameters a and b; that is, $\mu_i \sim \text{Beta}(a,b)$. We can vary these two parameters to increase both the variance and the left tail of $g(\mu)$ while keeping its mean constant. Crucially, these changes in $\{a,b\}$ also raise $\text{E}\mu^\alpha$.[12] Since $\frac{\partial v(t,s)}{\partial \text{E}\mu^\alpha} < 0$ and $\frac{\partial c^*(t)}{\partial \text{E}\mu^\alpha} < 0$, we have the following observation: Increasing the second and third moments of $g(\mu)$, while keeping its first moment constant, lowers welfare and fosters precautionary behaviour.

[12] I can only prove this last statement numerically, not analytically.

4. Summary

The two theoretical frameworks presented above convey the same simple message: the possibility of disastrous outcomes, however low their probability, motivates precautionary behaviour. That is, it becomes optimal to cut consumption to build a buffer of the consumption good.

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Appendix A. Proof of Proposition 1

I only prove the first result, for the others will follow immediately. By the chain rule:

$$\frac{\partial v(t,s)}{\partial \sigma} = \frac{s(t)^\alpha}{\alpha}\frac{\partial \theta(t)}{\partial \gamma}\frac{\partial \gamma}{\partial \sigma},$$

where $\frac{s(t)^\alpha}{\alpha} < 0$ and $\frac{\partial \gamma}{\partial \sigma} = \alpha(\alpha-1)\sigma > 0$. Because $\theta(t) > 0$, it can be shown that:

$$\frac{\partial \theta(t)}{\partial \gamma} > 0 \iff \varphi = 1 + e^{\frac{\gamma}{1-\alpha}(T-t)}.\left[\frac{\gamma(T-t)}{1-\alpha} - 1\right] > 0,$$

for all $\gamma \neq 0$.

Since $\frac{\partial \varphi}{\partial \gamma} = 0 \Leftrightarrow \gamma = 0$, $\frac{\partial \varphi}{\partial \gamma} > 0 \ \forall \gamma > 0$, and $\frac{\partial \varphi}{\partial \gamma} < 0 \ \forall \gamma < 0$, I have $\varphi > 0$ for all $\gamma \neq 0$. Hence, $\frac{\partial \theta(t)}{\partial \gamma} > 0$ and $\frac{\partial v(t,s)}{\partial \sigma} < 0$.

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